

Bond Duration and Convexity

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Bond duration and convexity are measures of the sensitivity of bond price to interest rate (i.e. yield) changes. Bond price is a function of time (t) and discount rate (k). The equation for bond price at time zero is the discounted value of expected future cash flow. The bond price equation in mathematical terms is...

$$P_0 = \sum_{t=1}^T [C_t \times (1+k)^{-t}] \quad (1)$$

Provided that estimated future cash flow does not change, bond price will change with the passage of time (t) and with changes in yield (k). Using a Taylor Series Approximation, the change in bond price, ΔP , over some small interval, Δt , is given by the following partial differential equation (PDE)...

$$\Delta P = \frac{\delta P}{\delta t} \Delta t + \frac{\delta P}{\delta k} \Delta k + \frac{1}{2} \frac{\delta^2 P}{\delta k^2} (\Delta k)^2 \quad (2)$$

The first term in the PDE is $\frac{\delta P}{\delta t} \Delta t$, which is the first derivative of the price equation with respect to time (t). This partial derivative measures the change in price that occurs with the passage of time while holding yield (k) constant. We are not interested in holding yield constant. We want to hold time constant and measure the change in price due to an instantaneous change in yield. For our purposes we can ignore this derivative.

The second term in the PDE is $\frac{\delta P}{\delta k} \Delta k$, which is the first derivative of the price equation with respect to yield (k). This partial derivative measures the first order change in price that occurs when yield changes while holding time (t) constant. We are interested in this derivative because this is what we want to know...how will bond price change with an instantaneous change in yield? This derivative is related to a bond's duration and is a linear measure of how bond price changes in response to interest rate changes. This derivative is always negative (Bond price increases with a decrease in interest rates. Bond price decreases with an increase in interest rates.

The third term in the PDE is $\frac{\delta^2 P}{\delta k^2} (\Delta k)^2$, which is the second derivative of the price equation with respect to yield (k) and measures how bond duration changes as interest rates change. A non-callable bond has positive convexity. Positive convexity means that a 100 basis point increase in rates will decrease bond price by less than what a 100 basis point decrease in rates will increase bond price. Convexity is a good thing that fixed income investors will pay for.

We now rewrite equation (2) by dividing both sides by bond price such that new PDE describes the percentage change in bond price. The new PDE becomes...

$$\frac{\Delta P}{P} = \frac{1}{P} \frac{\delta P}{\delta t} \Delta t + \frac{1}{P} \frac{\delta P}{\delta k} \Delta k + \frac{1}{2} \frac{1}{P} \frac{\delta^2 P}{\delta k^2} (\Delta k)^2 \quad (3)$$

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We can rewrite equation (1) by defining $\theta = (1 + k)$. The bond price equation becomes...

$$P_0 = \sum_{t=1}^T C_t \theta^{-t} \quad (4)$$

The first and second derivatives (respectively) of equation (4) with respect to θ is...

$$\frac{\delta P}{\delta \theta} = -\theta^{-1} \sum_{t=1}^T t C_t \theta^{-t} \quad (5)$$

$$\frac{\delta^2 P}{\delta \theta^2} = \theta^{-2} \sum_{t=1}^T (t^2 + t) C_t \theta^{-t} \quad (6)$$

We will define $\frac{1}{P} \frac{\delta P}{\delta \theta}$ as bond duration and $\frac{1}{P} \frac{\delta^2 P}{\delta \theta^2}$ as bond convexity. After making these substitutions into equation (3) and then multiplying by bond price, equation (3) becomes...

$$\Delta P = P \times Duration \times \Delta \theta + \frac{1}{2} \times P \times Convexity \times (\Delta \theta)^2 \quad (7)$$

A Hypothetical Case

Problem: A fixed income investor purchases a bond with a face value of \$1,000. The bond coupon rate is 8% and coupon payments are made semiannually. The bond has a contractual maturity of 10 years. The bond was purchased for \$1,229.40. This purchase price results in a yield-to-maturity of 6%. The investor wants to know what his investment would be worth if yields increased by 100 basis points (1.00%).

Inputs to the Duration and Convexity calculations:

Bond price	=	1,229.40
Face value	=	1,000.00
Coupon rate	=	8% (4% semiannually)
Current yield	=	6% (3% semiannually)
New yield	=	7% (3.5% semiannually)
Current theta	=	1.030
New theta	=	1.035
Change in theta	=	0.005

Duration and Convexity calculation:

Duration	=	-21.81 semiannual periods
Convexity	=	681.03 semiannual periods

Change in bond price:

$$\begin{aligned} \Delta P &= P \times Duration \times \Delta \theta + \frac{1}{2} \times P \times Convexity \times (\Delta \theta)^2 \\ \Delta P &= 1229.40 \times -21.81 \times 0.005 + 0.5 \times 1229.40 \times 681.03 \times 0.000025 \\ \Delta P &= -134.07 + 10.47 \\ \Delta P &= -123.60 \end{aligned}$$